Mathematical Journal of Okayama University

Volume 22, *Issue* 1 | 1980 | *Article* 15

JUNE 1980

On the Browder-Livesay invariant of free involutions on homology 3-spheres

Tomoyoshi Yoshida*

*Okayama University

Math. J. Okayama Univ. 22 (1980), 91—106

ON THE BROWDER-LIVESAY INVARIANT OF FREE INVOLUTIONS ON HOMOLOGY 3-SPHERES

Dedicated to Professor A. Komatu on his 70th birthday

TOMOYOSHI YOSHIDA

Introduction. Let M be a 3-dimensional homology sphere. Let T be a free involution on M. Let $\alpha(M, T)$ be the Browder-Livesay invariant, and let $\mu(M)$ be the μ-invariant of M. Then $\alpha(M, T)\in 8\boldsymbol{Z}$ and $\mu(M)\in 8\boldsymbol{Z}$ (mod 16). In §1, we shall prove that $\mu(M)=\alpha(M, T)$ (mod 16) (Theorem 1). This equation has been proved for Seifert homology spheres by W. D. Neumann and R. Raymond [11]. By this equation, in order to compute the μ invariant of those homology spheres which have free involution, it suffices to compute their Browder-Livesay invariant. For this purpose, in §2, we shall define 'signature invariants' $\sigma_{(m,n)}(V, T)$ for orientation preserving free involutions on 3-dimensional homology circles and coprime integer pairs (m, n) with m odd. In §3, by making use of these invariants, we shall give a formula of the Browder-Livesay invariant of some free involutions on homology spheres (Theorem 3 and its Corollary). This formula is a generalization of the formula given by W. D. Neumann [10] for Seifert homology spheres.

All the homology and cohomology groups in this paper have integer coefficients unless otherwise mentioned.

1. Browder-Livesay invariant and μ-invariant. For convenience, we shall work in the smooth category. All the manifolds will be compact and oriented. The boundary of X, ∂X, inherits its orientation from X. $-X$ will denote X with the opposite orientation. The unit n-disc will be denoted by D^n, the $(n-1)$-sphere, ∂D^n, by S^{n-1}, and the unit interval by I.

If X, Y are disjoint n-manifolds with X_0, Y_0 $(n-1)$ submanifolds of $\partial X, \partial Y$ respectively, and if $h: X_0\to Y_0$ is an orientation reversing diffeomorphism, then $X\cup_h Y$ will denote the quotient space $X\cup Y/(x\sim hx$ for all $x\in X_0)$. By the existence and uniqueness of collars, $X\cup_h Y$ has a natural structure as an n-manifold. If h comes from some natural identifications of X_0 and Y_0 with Z, say, we may write $X\cup_Z Y$ for $X\cup_h Y$.

A homology sphere (resp. $\boldsymbol{Z}_2$ homology sphere) is a closed 3-manifold M such that $H_*(M)\cong H_*(S^3)$ (resp. $H_*(M, \boldsymbol{Z}_2)\cong H_*(S^3, \boldsymbol{Z}_2)$). For a $\boldsymbol{Z}_2$ homology sphere M, Eells and Kuiper [2] have defined an invariant $\mu(M)\in 2\boldsymbol{Z}$ (mod 16) as follows: M bounds a 4-manifold Y such that $H_1(Y)$

Mathematical Journal of Okayama University, Vol. 22 [1980], Iss. 1, Art. 15

has no 2-torsion and the quadratic form of Y is even. Then, we set

$$\mu(M) = -\sigma(Y) \pmod{16},$$

where $\sigma(Y)$ is the signature of Y. This is well-defined, by Rohlin's theorem [12]. If M is a homology sphere, then the intersection form of Y is unimodular, and hence $\sigma(Y) \equiv 0 \pmod 8$, and the possible value of $\mu(M)$ is 8 or 0.

Let T be a free involution on a Z_2 homology sphere M. Then T preserves the orientation of M by Lefschetz fixed point theorem. The Browder-Livesay invariant $\alpha(M, T)$ is defined as follows ([1]): There is a decomposition of M such that $M = A \cup TA$, $A \cap TA = B$, where A is a 3-dimensional submanifold of M with $\partial A = B$, and B is an invariant closed surface in M (B is called a characteristic surface of (M, T)). A inherits its orientation from M. Put $\mathrm{K} = \mathrm{Ker}\ (H_1(B) \to H_1(A))$. Let $\langle\ ,\ \rangle$ be the bilinear form on K defined by $\langle x, y\rangle = x \cdot T_* y$ for $x, y \in \mathrm{K}$, where T_* is the homomorphism $H_1(B) \to H_1(B)$ induced by T, and dot denotes the intersection number. Then $\langle\ ,\ \rangle$ is a non-degenerate symmetric bilinear form on K of even type. Set

$$\alpha(M, T) = \sigma(\langle\ ,\ \rangle),$$

where σ is the signature of $\langle\ ,\ \rangle$. If M is a homology sphere, the bilinear from $\langle\ ,\ \rangle$ is unimodular, and hence $\alpha(M, T) \equiv 0 \pmod 8$.

The rest of this section will be devoted to the proof of the following

Theorem 1. *Let T be a free involution on a homology sphere M. Then $\mu(M) = \alpha(M, T) \pmod{16}$.*

Proof. Let M/T be the orbit space of M and let $p: M \to M/T$ be the covering projection. Let $\bar{C}$ be a simple closed curve in M/T which represents the generator of $H_1(M/T, Z_2) = Z_2$. Then $C = p^{-1}(\bar{C})$ is an invariant simple closed curve in M. Let U be an invariant closed tubular neighborhood of C in M. Then by Alexander duality theorem, it follows that $\overline{M-U}$ is a homology circle, $H_*(\overline{M-U}) = H_*(S^1)$, where $\overline{M-U}$ is the closure of $M-U$ in M. Now, both ∂U and $p(\partial U)$ are $S^1 \times S^1$. Let $\bar{D}$ be a simple closed curve in $p(\partial U)$ which is homologous to zero in $p(\overline{M-U})$. Then $p^{-1}(\bar{D})$ consists of two simple closed curves, say, D and its transformed image TD. Now there is an orientation preserving equivariant embedding $f: S^1 \times D^2 \to M$ such that $f(S^1 \times D^2) = U$, $f(S^1 \times 0) = C$ and $f(S^1 \times 1) = D$ $(\subset \partial \overline{M-U} = \partial U)$, where $S^1 \times D^2$ has the involution T defined by $T(x, y) = (-x, -y)$ for $(x, y) \in S^1 \times D^2$, and D^2 is regarded as the unit disc in the complex plane. Let $D^2\times D^2$ be the 4-disc with the involution T defined by $T(x, y)=(-x, y)$ $((x, y)\in D^2\times D^2)$. Let $g:\partial D^2\times D^2\to M$ be the equivariant embedding defined by $g(x, y)=f(x, xy)$ for $(x, y)\in\partial D^2\times D^2$. Form the manifold $M\times I\cup_g D^2\times D^2$, where g is considered as an embedding of $\partial D^2\times D^2$ to $M\times I$. Let M_0 be the manifold $\overline{M-U}\cup_{g|\partial D^2\times\partial D^2}D^2\times\partial D^2$, where $|$ denotes the restriction of the map. Then $\partial(M\times I\cup_g D^2\times D^2)=M\cup-M_0$. By Mayer-Vietoris theorem, it follows that M_0 is a homology sphere. Now M_0 has an involution T with fixed point set $C_0=0\times\partial D^2$. Let $M_1=M_0/T$ be the orbit space, and let $p_0:M_0\to M_1$ be the projection. Then it can be seen that any closed curve in M_1 intersecting transversely and non-vacuously with $C_1=p_0(C_0)$ lifts to a closed curve in M_0. Hence $p_{0*}:\pi_1(M_0)\to\pi_1(M_1)$ is onto. This implies that M_1 is a homology sphere. M_0 is considered as the 2-fold branched covering space of M_1 branched over the curve C_1. Let W be a 4-dimensional spin manifold which bounds M_1, $\partial W=M_1$, and is simply connected. Now $K_1=(M_1, C_1)$ is a knot in the homology sphere M_1. There is a Seifert surface of K_1, say S, in M_1 with $\partial S=C_1$ ([4]). Since S has a trivial normal bundle in M_1, there is an embedding of $S\times[-1, 1]$ into M_1 such that $S\times 0=S$. Now take two copies of W, say W_1 and W_2, and form the manifold $X=W_1\cup_{S\times[-1,1]}W_2$ from the disjoint union $W_1\cup W_2$ by identifying the point $(x, t)\in S\times[-1, 1]\subset\partial W_1$ with the point $(x, -t)\in S\times[-1, 1]\subset\partial W_2$. Then X has a natural involution T which is orientation preserving and interchanges W_1 with W_2 and W_2 with W_1. The fixed point set of T is $S=S\times 0$ and $\partial X\cap S=\partial S$. The orbit space X/T is W. ∂X is the 2-fold branched covering space of M_1 branched over the curve $\partial S=C_1$, and ∂X is equivariantly diffeomorphic to M_0. Let Y be the manifold $(M\times I\cup_g D^2\times D^2)\cup_{M_0}X$ obtained from the disjoint union $(M\times I\cup_g D^2\times D^2)\cup X$ by identifying the common boundary M_0. Then $\partial Y=M$ and Y has an orientation preserving involution T which restricts to the original involution on M. Now we compute $\alpha(M, T)$ and $\mu(M)$ by making use of Y.

First we consider the homology of $X=W_1\cup_{S\times I}W_2$. By Mayer-Vietoris theorem, there is the following exact sequence:

$$0\to H_2(W_1)\oplus H_2(W_2)\to H_2(X)\xrightarrow{\partial_*}H_1(S\times I)\to 0.$$

Let $\{\alpha_j\}_{j=1,\cdots,s}$ be a base of $H_2(W)$ and let $\{\alpha_j^{(i)}\}$ be the corresponding base of $H_2(W_i)$ $(i=1, 2)$, where $s=\operatorname{rank} H_2(W)$. Then $T_*\alpha_j^{(1)}=\alpha_j^{(2)}$ in $H_2(X)$, where T_* is the homomorphism induced by the involution T on X $(j=1, \cdots, s)$. Let $\{E_k\}_{k=1,\cdots,t}$ be curves in S which represent a base of $H_1(S)$, where $t=\operatorname{rank} H_1(S)$. Since each E_k is homologous to zero in

Mathematical Journal of Okayama University, Vol. 22 [1980], Iss. 1, Art. 15

$M_1=\partial W$, there exists a 2-chain G_k in a collar neighborhood $\partial W\times I$ such that $\partial G_k=E_k$. Let $G_k^{(i)}$ be the copies of G_k in W_i $(i=1, 2)$. Then $TG_k^{(1)}=G_k^{(2)}$ and $TG_k^{(2)}=G_k^{(1)}$. Now the chain $G_k^{(1)}-G_k^{(2)}$ is a cycle and represents a homology class β_k in $H_2(X)$ such that $\partial_*\beta_k=[E_k]\in H_1(S)$, where $[E_k]$ is the homology class represented by the curve E_k $(k=1, \cdots, t)$. Clearly $T_*\beta_k=-\beta_k$ in $H_2(X)$. By the above Mayer-Vietors tequence, $\{\alpha_j^{(1)}, \alpha_j^{(2)}, \beta_k\}$ $(j=1, \cdots, s$ and $k=1, \cdots, t)$ forms a base of $H_2(X)$. Next we consider $H_2(M\times I\cup_g D^2\times D^2)$. Let ξ be the homology class represented by the chain $C\times I\cup_{C\times 1}(0\times D^2)$, $\xi\in H_2(M\times I\cup_g D^2\times D^2, \partial)=H_2(M\times I\cup_g D^2\times D^2)$. Then $\xi\cdot\xi=1$ by the construction and ξ generates $H_2(M\times I\cup_g D^2\times D^2)$, and $T_*\xi=\xi$.

Now the set $\{\xi, \alpha_j^{(1)}, \alpha_j^{(2)}, \beta_k\}$ $(j=1, \cdots, s$ and $k=1, \cdots, t)$ is a base of $H_2(Y)$ such that

(1) $T_*\xi=\xi$, $T_*\alpha_j^{(1)}=\alpha_j^{(2)}$ and $T_*\beta_k=-\beta_k$,

(2) $\xi\cdot\xi=1$ and $\xi\cdot\alpha_j^{(1)}=\xi\cdot\alpha_j^{(2)}=\beta_k\cdot\alpha_j^{(1)}=\beta_k\cdot\alpha_j^{(2)}=\alpha_j^{(1)}\cdot\alpha_j^{(2)}=0$,

where dot denotes the intersection number ((2) is clear from the construction of ξ and β_k). Let $A^{(i)}$ be the intersection matrix $(\alpha_k^{(i)}\cdot\alpha_m^{(i)})$ $(i=1, 2$ and $k, m=1, \cdots, s)$ and let B be the intersection matrix $(\beta_k\cdot\beta_m)$ $(k, m=1, \cdots, t)$.

Assertion (1). $\alpha(M, T)=-\mathrm{sign}\, B$.

Proof. The fixed point set of the involution T on Y is the closed 2-surface $F=0\times D^2\cup_{c_0}S$. By Hirzebruch [5],

$$\alpha(M, T)=\mathrm{sign}\,(Y, T)-[F]\cdot[F],$$

where $\mathrm{sign}\,(Y, T)$ denotes the signature of the bilinear form $\langle\ ,\ \rangle$ on $H_2(Y)$ defind by $\langle x, y\rangle=x\cdot T_*y$ for $x, y\in H_2(Y)$, and $[F]$ is the homology class represented by F. With respect to the above base of $H_2(Y)$, the bilinear form $\langle\ ,\ \rangle$ is represented by the matrix

$$\langle 1\rangle+\begin{bmatrix}0 & A\\ A & 0\end{bmatrix}+\langle -B\rangle,$$

where $A=A^{(1)}=A^{(2)}$. Hence $\mathrm{sign}\,(Y, T)=1-\mathrm{sign}\, B$. Now $[F]\cdot[F]=1$, and the Assertion (1) follows.

Assertion (2). $\mu(M)=-\mathrm{sign}\, B \pmod{16}$.

Proof. With respect to the above base, the intersection form on $H_2(Y)$ is represented by that matrix

$$\langle 1\rangle+A+A+B,$$

where $A=A^{(1)}=A^{(2)}$. Hence $\sigma(Y)=1+2\operatorname{sign} A+\operatorname{sign} B$. Since $\partial W=M_1$ is a homology sphere and W is a spin manifold, A is a unimodular symmetric matrix with even diagonal entries. Therefore $\operatorname{sign} A\equiv 0 \pmod 8$, and we obtain $\sigma(Y)=1+\operatorname{sign} B \pmod{16}$. Now the class ξ is represented (as a relative cycle) by a disc $C\times I\cup_{C\times 1}(0\times D^2)$ with boundary $C\subset M$, and ξ is characteristic for the quadratic form of Y, that is, $\xi\cdot z=z\cdot z \pmod 2$ for every $z\in H_2(Y)$. Let K be the knot (M, C) in the homology sphere M. Then Gordon [4] has shown that $\mu(M)=\xi\cdot\xi-\sigma(Y)-c(K) \pmod{16}$, where $c(K)$ is the Arf invariant of the knot $K=(M, C)$. Hence $\mu(M)=-\operatorname{sign} B-c(K) \pmod{16}$. Now it suffices to prove that $c(K)=0 \pmod{16}$. Let $\Delta_K(t)$ be the Alexander polynomial of K. Then $c(K)=(\Delta_K(-1))^2-1 \pmod{16}$ ([7]). Now the knot $K=(M, C)$ and the knot $K_0=(M_0, C_0)$ have the same knot complement, so that $\Delta_K(t)=\Delta_{K_0}(t)$. Since M_0 is the 2 fold branched covering space of M_1 branched over the curve C_1, $\Delta_{K_0}(t^2)=\Delta_{K_1}(t)\Delta_{K_1}(t^{-1})$, where $K_1=(M_1, C_1)$ is the knot in the homology sphere M_1 ([3]). Hence $\Delta_K(-1)=\Delta_{K_0}(-1)=\Delta_{K_1}(\sqrt{-1})\Delta_{K_1}(-\sqrt{-1})$. Since the Alexander polynomial of any knot is a reciprocal polynomial, it can be seen that $\Delta_{K_1}(\sqrt{-1})\Delta_{K_1}(-\sqrt{-1})$ is a square of an odd integer, say, $(2d+1)^2$ (note that $\Delta_{K_1}(-1)\equiv 1 \pmod 2$). Therefore $c(K)=(\Delta_K(-1))^2-1=8d(d+1)(2d^2+2d+1)\equiv 0 \pmod{16}$. This proves Assertion (2), and therefore Theorem 1.

Remark 1. Let $K_1=(M_1, C_1)$ be the knot in the above proof. Then it is well known that sign B in the above proof is equal to the so-called signature of the knot K_1, $\sigma(K_1)$. Hence we have proved that $\alpha(M, T)=-\sigma(K_1)$.

Remark 2. The equation in Theorem 1 has been proved for Seifert homology spheres by W. D. Neumann and F. Raymond [11]. Many examples of free involutions on homology spheres which cannot be embedded in S^1 action have been constructed by C. McA. Gordon [4]. The same argument in the above proof gives only $\mu(M)=\alpha(M, T) \pmod 4$ for Z_2 homology spheres with free involution. In general, the equality $\mu(M)=\alpha(M, T) \pmod{16}$ does not hold for Z_2 homology spheres with free involution. As an example, let $L(3, 1)$ be the standard Lens space and let M be the connected sum $L(3, 1)\# L(3, 1)$. Then M has a natural free involution with S^2 as the characteristic surface. Hence $\alpha(M, T)=0$. On the other hand, $\mu(M)=2\mu(L(3, 1))=4 \pmod{16}$.

2. Signatures of orientaion preserving involutions on homology

Mathematical Journal of Okayama University, Vol. 22 [1980], Iss. 1, Art. 15

circles. Let V be a homology circle, that is, a compact 3-dimensional manifold with boundary $S^1 \times S^1$ such that $H_*(V)=H_*(S^1)$. Let T be an orientation preserving free involution on V. Let V/T be the orbit space and let $p: V \to V/T$ be the covering projection.

Lemma 2.1. *V/T is a homology circle.*

Proof. Let S^∞ be the infinite dimensional sphere with the antipodal free involution T. Then $S^\infty/T=P^\infty$ is the infinite dimensional real projective space and $S^\infty \to P^\infty$ is the universal Z_2 bundle. Let $S^\infty \times_T V$ be the orbit space of $S^\infty \times V$ by the diagonal action of Z_2. Since the involution on V is free and S^∞ is contractible, $S^\infty \times_T V$ is homotopy equivalent to V/T. The projection to the first factor $S^\infty \times_T V \to P^\infty$ is a fibre bundle with fibre V. Considering the cohomology Serre spectral sequence of this fibre bundle, it follows that $p^*: H^*(V/T) \to H^*(V)$ is injective and $p^*(H^1(V/T))=2\boldsymbol{Z} \subset \boldsymbol{Z}=H^1(V)$. This proves Lemma 2. 1.

Definition 2.1. *A pair of characteristic curves in* ∂V is defined as a pair of oriented simple closed curves (C, D) in V such that (i) $TC=C$ and $D \cap TD=\emptyset$, and (ii) $C \cap D$ consists of a single point and $C \cdot D=1$, where dot denotes the intersection number.

Lemma 2.2. *For $i=1, 2$, let V_i be a homology circle with free orientation preserving involution T_i. Let (C_i, D_i) be a pair of characteristic curves in ∂V_i. Then there is an orientation reversing equivariant diffeomorphism $h: \partial V_1 \to \partial V_2$ such that h maps C_1 onto C_2 orientation preservingly and D_1 onto D_2 orientation reversingly.*

Proof. Let $p_i: V_i \to V_i/T_i$ be the covering projection $(i=1, 2)$. Then $(p_i(C_i), p_i(D_i))$ is a pair of simple closed curves in $\partial(V_i/T_i)$ whose homology classes generate $H_1(\partial(V_i/T_i))$. Since $\partial(V_i/T_i)$ is $S^1 \times S^1$, there is a diffeomorphism $\bar{h}: \partial(V_1/T_1) \to \partial(V_2/T_2)$ such that $\bar{h}$ maps $p_1(C_1)$ onto $p_2(D_2)$ orientation preservingly and $p_1(D_1)$ onto $p_2(D_2)$ orientation reversingly. Now it suffices to take as h the map which covers $\bar{h}$.

Definition 2.2. A pair of characteristic curves in ∂V, (C, D), is called of *type* (m, n) if $[C]=m\lambda$ and $[D]=n\lambda$ in $H_1(V)$, where λ is a base of $H_1(V)=\boldsymbol{Z}$ and $[C]$ and $[D]$ are represented classes. We always choose λ so that $m>0$.

The integer pairs (m, n) in the above definition are coprime pairs if $n \neq 0$.

Lemma 2.3. *Let (C, D) be a pair of characteristic curves in ∂V of type (m, n). Then m is an odd integer.*

Proof. Let $p: V \to V/T$ be the covering projection. Then $p(C)$ is a simple closed curve in $\partial(V/T)$. Let $\bar{\lambda}$ be the base of $H_1(V/T)$ such that $p_*(\lambda) = 2\bar{\lambda}$, where p_* is the induced homomorphism. Then $[p(C)] = m\bar{\lambda}$ in $H_1(V/T)$. The map p is a covering map associated to the composite homomorphism $\rho: \pi_1(V/T) \to H_1(V/T) = \boldsymbol{Z} \to \boldsymbol{Z}/2\boldsymbol{Z}$, where the last map is the mod 2 reduction. Since $C \to p(C)$ is a non-trivial covering, $\rho(p(C)) \equiv m \not\equiv 0 \pmod 2$. Hence m is odd.

First we consider a pair of characteristic curves in ∂V of type $(1, 0)$, (C, D). The curve D is null-homologous to zero in V.

Lemma 2.4. *Let (C, D) be a pair of characteristic curves in ∂V of type $(1, 0)$. Then there is a connected surface B in V such that $B \cap TB = \emptyset$, $\partial B = B \cap \partial V = D$ and $H_1(B) \to H_1(V)$ is a zero-homomorphism, where the last map is induced by the inclusion.*

Proof. In the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(V) & \xrightarrow{p_*} & \pi_1(V/T) & \longrightarrow & \boldsymbol{Z}_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & H_1(V) & \xrightarrow{p_*} & H_1(V/T) & \longrightarrow & \boldsymbol{Z}_2 & \longrightarrow & 1
\end{array},$$

the upper line is a non-trivial extension of $\pi_1(V)$ by $\boldsymbol{Z}_2$, and the vertical maps are the abelianizations of the groups, and the lower line is equivalent to the sequence $\boldsymbol{Z} \xrightarrow{2} \boldsymbol{Z} \to \boldsymbol{Z}_2$. Since S^1 is $K(\boldsymbol{Z}, 1)$, there is the following pull back diagram of the covering spaces corresponding to the above diagram

$$\begin{array}{ccc}
V & \xrightarrow{p} & V/T \\
\downarrow & & \downarrow f \\
S^1 & \xrightarrow{2} & S^1
\end{array},$$

where $S^1 \xrightarrow{2} S^1$ is the usual 2-fold covering and f is a map representing a generator of $H^1(V/T)$. We may assume that f is t-regular at $1 \in S^1$. $f^{-1}(1)$ is a surface in V/T such that $f^{-1}(1) \cap \partial(V/T) = \partial(f^{-1}(1))$ is a simple closed curve in $\partial(V/T)$. Since both $p(D)$ and $\partial(f^{-1}(1))$ are nul-homologous in V/T and $\partial(V/T) = S^1 \times S^1$, $p(D)$ and $f^{-1}(1)$ are homotopic, and hence isotopic in $\partial(V/T)$. Therefore we may assume that $\partial f^{-1}(1)=p(D)$. Let B be the surface which is the connected component of $p^{-1}(f^{-1}(1))$ containing D as its boundary. Then B satisfies the conditions requested.

Let B be such a surface as in Lemma 2.4. Then $B\cup TB$ decomposes V as $V=A\cup TA$, $A\cap TA=B\cup TB$, where A is a 3-dimensional submanifold of V. By Mayer-Vietoris theorem, we see that $H_1(B\cup TB)=H_1(A)\oplus H_1(TA)$. Let K be the kernel of the homomorphism $H_1(B\cup TB)\to H_1(A)$ induced by the inclusion. Then $H_1(B\cup TB)=K+T_*K$, where $T_*: H_1(B\cup TB)\to H_1(B\cup TB)$ is induced by T. Let $\langle\ ,\ \rangle$ be the bilinear form on K defined by $\langle x, y\rangle=x\cdot T_*y$ for $x, y\in K$. Then $\langle\ ,\ \rangle$ is a symmetric bilinear form of even type on K. Let $\sigma_{(C,D,B)}(V, T)$ be the signature of this symmetric bilinear form.

Now, let T_0 be the involution on $S^1\times D^2$ defined by $T_0(x, y)=(-x, y)$ for $(x, y)\in S^1\times D^2$. By Lemma 2.2, there is an orientation reversing equivariant diffeomorphism $h:\partial V\to\partial(S^1\times D^2)$ such that h maps D onto the curve $E=\{(x, x)\mid x\in S^1\}\subset\partial(S^1\times D^2)$. Then $M_h=V\cup_h S^1\times D^2$ is a homology sphere with free involution T.

Lemma 2.5. *Let M_h be as above. Then $\sigma_{(C,D,B)}(V, T)=\alpha(M_h, T)$, where α is the Browder Livesay invariant.*

Proof. There is an invariant annulas $S^1\times I$ in $S^1\times D^2$ such that $S^1\times 0=E$ and $S^1\times 1=T_0E$. Put $\bar{B}=B\cup_E S^1\times I\cup_{T_0E}TB$. Then $\bar{B}$ is a characteristic surface of M_h, and M_h decomposes as $M_h=\bar{A}\cup T\bar{A}$, $\bar{A}\cap T\bar{A}=\bar{B}$, where $\bar{A}$ is a submanifold of M_h such that $\bar{A}\cap V=A$ (A is as above). Clearly $H_1(\bar{B})=H_1(B\cup TB)$ and $\mathrm{Ker}(H_1(\bar{B})\to H_1(\bar{A}))=\mathrm{Ker}(H_1(B\cup TB)\to H_1(A))=K$. By the definition of $\alpha(M_h, T)$ and $\sigma_{(C,D,B)}(V, T)$, Lemma 2.5 follows.

By the above lemma, $\sigma_{(C,D,B)}(V, T)$ does not depend on the choice of B. Now let (C', D') be another pair of characteristic curves in ∂V of type $(1, 0)$. Let $\bar{h}:\partial(V/T)\to\partial(S^1\times D^2/T_0)$ be the map induced by the above equivariant map h. Since $p(D)$ and $p(D')$ are homotopic, and hence isotopic in $\partial V/T$, there is a diffeomorphism $\bar{h}':\partial V/T\to\partial(S^1\times D^2/T_0)$ such that $\bar{h}'$ is isotopic to $\bar{h}$ and $\bar{h}'$ maps $p(D')$ onto $p_0(E)$, where p and p_0 are the covering projections of V and $S^1\times D^2$ respectively. Let h' be the map which covers $\bar{h}'$. Then h' is equivariantly isotopic to h and $h'(D)=E$. Now form the manifold M_h, as above. Then $\alpha(M_h, T)=\alpha(M_{h'}, T)$. This shows that $\sigma_{(C,D,B)}(V, T)$ does not depend on the particular choice of (C, D) of type $(1, 0)$. Hence the following definition is possible.

Definition 2.3. Let V be a homology circle with orientation preserving involution T. Define $\sigma(V, T)=\sigma_{(C,D,B)}(V, T)$, where (C, D, B) is as above.

Next we consider a pair of characteristic curves in ∂V of type (m, n), (C, D), where m is odd>0 and $n\neq 0$. Since $D\cup TD$ is a characteristic submanifold of ∂V, the relative transversality theorem shows that there is an invariant surface B in V such that $B=TB$ and $\partial B=D\cup TD$, and B decomposes V as $V=A\cup TA$, $A\cap TA=B$, where A is a 3-dimensional submanifold in V. Let K be the kernel of the homomorphism $H_1(B)\to H_1(A)$ induced by the inclusion. Let $\langle\ ,\ \rangle$ be the bilinear form on K defined by $\langle x, y\rangle=x\cdot T_*y$ for $x, y\in K$. Let $\sigma_{(C,D,B)}(V)$ be the signature of this symmetric bilinear form. First we consider the case with $(V, T)=(S^1\times D^2, T_0)$, where T_0 is as before.

Lemma 2.6. *Let (C, D) be a pair of characteristic curves in $\partial(S^1\times D^2)$ of type (m, n), where $n\neq 0$. Let B be a characteristic surface of $S^1\times D^2$ such that $\partial B=D\cup TD$. Then $\sigma_{(C,D,B)}(S^1\times D^2, T_0)=\alpha(L(n, m), T)$, where α is the Browder-Livesay invariant, $L(n, m)$ is the standard Lens space and T is the involution on $L(n, m)$ with orbit space $L(2n, m)$.*

Proof. Let $f:\partial(S^1\times D^2)\to\partial(S^1\times D^2)$ be an equivariant diffeomorphism such that f maps $S^1\times 1$ onto C orientation preservingly and $-(1\times S)$ onto D orientation reversingly. Since (C, D) is of type (m, n), the manifold $S^1\times D^2\cup_f S^1\times D^2$ is equivariantly diffeomorphic to $L(n, m)$ with the above involution T. Now $\bar{B}=(1\times D^2)\cup_{(1\times S^1)}B\cup_{(-1\times S^1)}(-1\times D^2)$ is a characteristic surface of $L(n, m)$, and there is a 3-dimensional submanifold $\bar{A}$ of $L(n, m)$ such that $L(n, m)=\bar{A}\cup T\bar{A}$, $\bar{A}\cap T\bar{A}=\bar{B}$ and $\bar{A}=(\bar{A}\cap(S^1\times D^2))\cup_f(\bar{A}\cap(S^1\times D^2))=I\times D^2\cup_f A$, where A is as above. Clearly Ker $(H_1(\bar{B})\to H_1(\bar{A}))=$ Ker $(H_1(B)\to H_1(A))$. By the definition of $\sigma_{(C,D,B)}(S^1\times D^2)$, we obtain the lemma.

Lemma 2.7. *Let V be a homology circle with orientation preserving free involution T. Let (C, D) be a pair of characteristic curves in ∂V of type (m, n), where m is odd>0 and $n\neq 0$. Let B be a characteristic surface in V such that $\partial B=D\cup TD$. Then*

$$\sigma_{(C,D,B)}(V, T)=\sigma(V, T)+\alpha(L(n, m), T),$$

Mathematical Journal of Okayama University, Vol. 22 [1980], Iss. 1, Art. 15

where $L(n, m)$ is the standard Lens space and T is the free involution on $L(n, m)$ with orbit space $L(2n, m)$.

Proof. Let $(S^1 \times D^2, T_0)$ be as before. Let $h: \partial(S^1 \times D^2) \rightarrow \partial V$ be an orientation reversing equivariant diffeomorphism such that h maps $S^1 \times 1$ onto C orientation preservingly and $-(1 \times S^1)$ onto D orientation reversingly. Let M be the manifold $(S^1 \times D^2) \cup_f V$. M has a free involution T, and by the same argument in the proof of Lemma 2. 6 we obtain $\alpha(M, T) = \sigma_{(C,D,B)}(V, T)$. Now let (C', D') be a pair of characteristic curves in ∂V of type $(1, 0)$. Let B' be a surface in V such that $B' \cap TB' = \emptyset$, $\partial B' = B' \cap \partial V = D'$ and $H_1(B') \rightarrow H_1(V)$ is a zero-homomorphism, where the last map is induced by inclusion (see Lemma 2. 4). Put $C_0 = h^{-1}(C')$ and $D_0 = h^{-1}(D')$. Then it can be seen that (C_0, D_0) is a pair of characteristic curves in $\partial(S^1 \times D^2)$ of type $(-r, n)$, where r is an odd integer such that $mr \equiv -1 \pmod n$. Let B_0 be a characteristic surface in $S^1 \times D^2$ such that $\partial B_0 = D_0 \cup T_0 D_0$. Then $\bar{B} = B_0 \cup B' \cup TB'$ is a characteristic surface in M. There is a 3-dimensional submanifold $\bar{A}$ with boundary $\partial \bar{A} = \bar{B}$ in M such that $M = \bar{A} \cup T\bar{A}$ and $\bar{A} \cap T\bar{A} = \bar{B}$. Put $A_0 = \bar{A} \cap (S^1 \times D^2)$ and $A' = \bar{A} \cap V$. Then $A_0 \cup TA_0 = S^1 \times D^2$, $A_0 \cap TA_0 = B_0$, $A' \cup TA' = V$ and $A' \cap TA' = B' \cup TB'$. Now it follows that
$\mathrm{Ker}\,(H_1(\bar{B}) \rightarrow H_1(\bar{A})) = \mathrm{Ker}\,(H_1(B_0) \rightarrow H_1(A_0)) + \mathrm{Ker}\,(H_1(B' \cup TB') \rightarrow H_1(A'))$.
Hence we see that $\alpha(M, T) = \sigma_{(C_0, D_0, B_0)}(S^1 \times D^2, T_0) + \sigma_{(C', D', B')}(V, T)$. By definition, $\sigma_{(C', D', B')}(V, T) = \sigma(V, T)$. By Lemma 2. 6, $\sigma_{(C_0, D_0, B_0)}(S^1 \times D^2, T_0) = \alpha(L(n, -r), T)$, where T is the free involution on $L(n, -r)$ with orbit space $L(2n, -r)$. Since $mr \equiv -1 \pmod n$, $L(n, -r)$ is equivariantly and orientation preservingly diffeomorphic to $L(n, m)$. Hence $\alpha(L(n, -r), T) = \alpha(L(n, m), T)$. Therefore we obtain $\alpha(M, T) = \sigma(V, T) + \alpha(L(n, m), T)$. This implies that $\sigma_{(C,D,B)}(V, T) = \sigma(V, T) + \alpha(L(n, m), T)$.

Definition 2. 3. For a homology circle V with orientation preserving free involution T, we define $\sigma_{(m,n)}(V, T) = \sigma_{(C,D,B)}(V, T)$, where (C, D) is a pair of characteristic curves in ∂V of type (m, n) and B is a characteristic surface in V such that $\partial B = D \cup TD$.

Following [11], we use the following notation: $\mathrm{c}(p, q)$ is defined for coprime integer pairs (p, q) with p odd by the recursions

$$\mathrm{c}(p, \pm 1) = 0$$
$$\mathrm{c}(p, -q) = \mathrm{c}(-p, q) = -\mathrm{c}(p, q)$$
$$\mathrm{c}(p, p+q) = \mathrm{c}(p, q) + \mathrm{sign}\,(q(p+q))$$
$$\mathrm{c}(p+2q, q) = \mathrm{c}(p, q).$$

In [11], it is claimed that $c(p, q)=\alpha(L(q, p), T)$, where T is the free involution on $L(q, p)$ with orbit space $L(2q, p)$.

Now, by Lemmas 2. 7 and 2. 5, we readily obtain the following

Theorem 2. *For a homology circle V with orientation preserving free involution T and for a coprime integer pair (m, n) with m odd,*

$$\sigma_{(m,n)}(V, T)=\sigma(V, T)+c(m, n).$$

Moreover, $\sigma(V, T)\equiv 0 \pmod 8$.

Proposition 1. *Let V_1 and V_2 be homology circles with free orientation preserving involutions T_1 and T_2, respectively. Let $h:\partial V_1 \to \partial V_2$ be an orientation reversing equivariant diffeomorphism such that $M=V_1\cup_h V_2$ is a homology sphere. Then*

$$\alpha(M, T)=\sigma(V_1, T_1)+\sigma(V_2, T_2).$$

Proof. Let (C, D) be a pair of characteristic curves in ∂V_1 of type $(1, 0)$. Then as M is a homology sphere, $(h(C), h(D))$ is a pair of characteristic curves in ∂V_2 of type $(m, \pm 1)$ for some odd integer m. By the same argument in the proof of Lemma 2. 7, we see that $\alpha(M, T)= \sigma(V_1, T_1)+\sigma_{(m,\pm 1)}(V_2, T_2)$. Now $\sigma_{(m,\pm 1)}(V_2, T_2)=\sigma(V_2, T_2)+c(m, \pm 1)= \sigma(V_2, T_2)$.

Some examples. (1) Let T be the antipodal involution on S^3. Let K be a knot in S^3 such that $TK=K$. Let N be an invariant closed tubular neighborhood of K in S^3. Let $V(K)$ be the closure of the complement of N, $V(K)=\overline{S^3-N}$. Then $V(K)$ is a homology circle with free involution T.

Proposition 2. $\sigma(V(K), T)=0$.

Proof. Since $S^3=V(K)\cup N$, $\alpha(S^3, T)=\sigma(V(K), T)+\sigma(N, T)$ by Proposition 1. Now $\alpha(S^3, T)=\sigma(N, T)=0$.

(2) Let K be a knot in S^3 such that $\Delta(-1)=\pm 1$, where $\Delta(t)$ is the Alexander polynomial of K. Let M be the 2-fold branched covering space of S^3 branched over K. Then M is a homology sphere with involution. Let N be an invariant closed tubular neighborhood of K in M. Let V_K be the cosure of the complement of N, $V_K=\overline{M-N}$. Then V_K is a homology circle with free involution T.

Mathematical Journal of Okayama University, Vol. 22 [1980], Iss. 1, Art. 15

Proposition 3. *$\sigma(V_K, T) = -\sigma(K)$, where $\sigma(K)$ is the signature of the knot K.*

Proof. Let $(S^1 \times D^2, T_0)$ be as before. Let $h : \partial(S^1 \times D^2) \to \partial V_K$ be an orientation reversing equivariant diffeomorphism such that $(h(S^1 \times 1), h(1 \times S^1))$ is a pair of characteristic curves in ∂V of type $(m, \pm 1)$ for some odd integer m. Then $H = S^1 \times D^2 \cup_h V$ is a homology sphere with free involution T. By Proposition 1, $\alpha(H, T) = \sigma(V, T)$. Now the process of construcing H from the knot K in S^3 is just the converse of the process of constructing the knot $K_1 = (M_1, C_1)$ in the proof of Theorem 1. By Remark 1, $\alpha(H, T) = -\sigma(K)$. Hence $\sigma(V_K, T) = -\sigma(K)$.

3. Browder-Livesay invariants of some homology spheres. In this section, first we consider the homology circles with free involution constructed as follows :

Let V_i $(i = 1, \cdots, k)$ be a homology circle with orientation preserving free involution T_i. Let (C_i, D_i) be a pair of characteristic curves in ∂V_i of type (m_i, n_i) $(i = 1, \cdots, k)$. Let T_0 be the free involution on $S^1 \times D^2$ defined by $T_0(x, y) = (-x, y)$ for $(x, y) \in S^1 \times D^2$. Let $\{p_i\}_{i=1,\cdots,k}$ be k distinct points of ∂D^2 and let E_i be an arc in ∂D^2 containing p_i in its interior such that $E_i \cap E_j = \emptyset$ for each $i \neq j$ $(i, j = 1, \cdots, k)$. Let $V((m_1, n_1), \cdots, (m_k, n_k))$ be the 3-manifold obtained from the disjoint union $(S^1 \times D^2) \cup (\bigcup_{i=1}^{k} V_i)$ by identifying each $S^1 \times E_i$ with an invariant annular neighborhood of C_i in ∂V_i by an equivariant diffeomorphism f_i which maps $S^1 \times p_i$ onto C_i in an orientation preserving manner and reverses orientation on a transverse arc.

Lemma 3.1. *If m_i and m_j are coprime for each $i \neq j$ $(i, j = 1, \cdots, k)$, then $V((m_1, n_1), \cdots, (m_k, n_k))$ is a homology circle.*

Proof. There are integers $r_1, \cdots, r_k$ such that $\sum m_1 \cdots m_{i-1} r_i m_{i+1} \cdots m_k = 1$. Let λ_i be a generator of $H_1(V_i)$ such that $[C_i] = m_i \lambda_i$ $(i = 1, \cdots, k)$. By Mayer Vietoris theorem, we see that $r_1\lambda_1 + \cdots + r_k\lambda_k$ generates $H_1(V((m_1, n_1), \cdots, (m_k, n_k)) = \boldsymbol{Z}$.

Since each f_i is equivariant, $V((m_1, n_1), \cdots, (m_k, n_k))$ has a free involution T.

Theorem 3. *Let $V = V((m_1, n_1), \cdots, (m_k, n_k))$ be the above manifold, where m_i and m_j are coprime for each $i \neq j$ $(i, j = 1, \cdots, k)$. Then*

$$\sigma(V, T) = \sum_{i=1}^{k} (\sigma(V_i, T_i) + \mathrm{c}(m_i, n_i) + \operatorname{sign} n_i)$$
$$-\mathrm{c}\left(m_1 \cdots m_k, m_1 \cdots m_k \sum_{i=1}^{k} \frac{n_i}{m_i}\right) - \operatorname{sign}\left(\sum_{i=1}^{k} \frac{n_i}{m_i}\right),$$

where σ *and* c *are as in* §2, *and* $\mathrm{c}(m, 0) = \operatorname{sign} 0 = 0$.

Corollary. *Let* h *be an orientation reversing equivariant diffeomorphism from* $\partial(S^1 \times D^2)$ *to* ∂V *such that* $(S^1 \times D^2) \cup_h V$ *is a homology sphere with free involution* T, *where* V *is as in Theorem* 3. *Then* $\alpha((S^1 \times D^2) \cup_h V, T)$ *is given by the right hand term of the equation in Theorem* 3.

Proof. This is straightforward from Proposition 1 in §2 and Theorem 3.

Remark. When $(V_i, T_i) = (S^1 \times D^2, T_0)$ for $i = 1, \cdots, k$ in the above construction, it can be seen that $V = V((m_1, n_1), \cdots, (m_k, n_k))$ has an S^1 action which extends the involution. Moreover in the case that m_i and m_j are coprime for each $i \neq j$ $(i, j = 1, \cdots, k)$ and $m_1 \cdots m_k \left(\sum_{i=1}^{k} \frac{n_i}{m_i}\right) = \pm 1$, by a suitable choice of h in the above Corollary, the resulting manifold $S^1 \times D^2 \cup_h V = M$ may be endowed with S^1 action which extends the involution, that is, this manifold is a Seifert manifold and $(m_1, n_1), \cdots, (m_k, n_k)$ are its Seifert invariants. In this case the above formula of $\alpha(V, T)$ gives the formula of W. D. Neumann [10],

$$\alpha(M, T) = \sum_{i=1}^{k} (\mathrm{c}(m_i, n_i) + \operatorname{sign} n_i) - \operatorname{sign}\left(\sum_{i=1}^{k} \frac{n_i}{m_i}\right).$$

Hence the above formula is a generalization of Neumann's formula for the Browder-Livesay invariants of Seifert homology spheres.

Proof of Theorem 3. We prove Theorem 3 by the induction on k. When $k = 1$, Theorem 3 is trivial since V is equivariantly diffeomorphic to V_1 and $\operatorname{sign} n_1 = \operatorname{sign} n_1/m_1$ $(m_1 > 0)$. Next we consider the case with $k = 2$. The formula which we must prove is

$$\sigma(V, T) = \sigma(V_1, T_1) + \sigma(V_2, T_2) + \mathrm{c}(m_1, n_1) + \mathrm{c}(m_2, n_2) - \mathrm{c}(m_1 m_2, m_1 n_2 + m_2 n_1)$$
$$+ \operatorname{sign} n_1 + \operatorname{sign} n_2 - \operatorname{sign}(n_1/m_1 + n_2/m_2).$$

Let B_i be a characteristic surface of V_i such that $\partial B_i = D_i \cup TD_i$ $(i = 1, 2)$. If $n_i = 0$, then we take $B \cup TB$ as B_i where B is a surface in V_i satisfying the conditions of Lemma 2. 4. We may assume that $f_i^{-1}(D_i) = 1 \times E_i \subset \partial(S^1 \times D^2)$ and $f_i^{-1}(TD_i) = -1 \times E_i$. Then $\bar{B} = B_1 \cup B_2 \cup 1 \times D^2 \cup -1 \times D^2$

Mathematical Journal of Okayama University, Vol. 22 [1980], Iss. 1, Art. 15

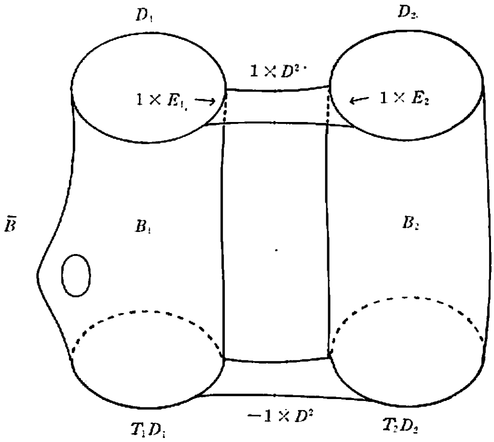

Figure 1

(where $B_i \subset V_i$ ($i=1, 2$) and $\{\pm 1\}\times D^2 \subset S^1\times D^2$) is a characteristic surface of V (Figure 1). Put $\overline{D_1\cup D_2 \cup 1\times\partial D^2 - 1\times(E_1\cup E_2)} = D$. Let r_1 and r_2 be integers such that $r_1m_2 + r_2m_1 = 1$. Then $\lambda = r_1\lambda_1 + r_2\lambda_2$ generates $H_1(V) = \boldsymbol{Z}$, and $\lambda_1 = m_2\lambda$ and $\lambda_2 = m_1\lambda$. Now $[D_1] = n_1\lambda_1 = n_1m_2\lambda$ and $[D_2] = n_2\lambda_2 = n_2m_1\lambda$ in $H_1(V)$. Therefore $[D] = (n_1m_2+n_1m_1)\lambda$ in $H_1(V)$. Let C be the curve $S^1\times p \subset \partial(S^1\times D^2)$, where $p\in D^2 - \bigcup_{i=1}^{k} E_k$. Then $C\subset\partial V$. Clearly $C\cap D$ consists of a single point, and $[C] = m_1\lambda_1 = m_1\lambda_2 = m_1m_2\lambda$ in $H_1(V)$. Hence (C, D) is a pair of characteristic curves in ∂V of type $(m_1m_2, m_1n_2+m_2n_1)$. Therefore $\sigma_{(C,D,B)}(V, T) = \sigma_{(m_1m_2, m_1n_2+m_2n_1)}(V, T)$.

Assertion (1). $\sigma_{(C,D,B)}(V, T) = \sigma_{(C_1,D_1,B_1)}(V_1, T_1) + \sigma_{(C_2,D_2,B_2)}(V_2, T_2) + \varepsilon$, *where* $\varepsilon = \pm 1$ *if* $n_1n_2 \neq 0$, *and* $\varepsilon = 0$ *if* $n_1n_2 = 0$.

Proof. If $n_i \neq 0$, then the curve D_i is not homologous to zero in V_i and there is a connected component in B_i with boundary $D_i\cup TD_i$, and if $n_i = 0$, then $B_i = B\cup TB$ where B is a surface such that $\partial B = D_i$ and $\partial TB = TD_i$ ($i=1, 2$). Hence by Mayer-Vietoris theorem, it follows that rank $H_1(B) =$ rank $H_1(B_1) +$ rank $H_1(B_2) + 2$ if $n_1n_2\neq 0$, and $H_1(B)\cong H_1(B_1)+H_1(B_2)$ if $n_1n_2 = 0$. Let $\bar{A}$ be a 3-dimensional submanifold of V such that $V = \bar{A}\cup T\bar{A}$, $\bar{A}\cap T\bar{A} = \bar{B}$. Put $A_1 = \bar{A}\cap V_1$ and $A_2 = \bar{A}\cap V_2$. Then $A_i\cup TA_i = V_i$ and $A_i\cap TA_i = B_i$ ($i=1, 2$). Put $K = \mathrm{Ker}\,(H_1(\bar{B})\to H_1(\bar{A}))$ and put $K_i = \mathrm{Ker}\,(H_1(B_i)\to H_i(A_i))$ ($i=1, 2$), where all the homomorphisms are induced by the inclusions. Then we see that rank $K =$ rank $K_1 +$ rank $K_2 + 1$ if $n_1n_2\neq 0$, and $K = K_1 + K_2$ if $n_1n_2 = 0$. Now $K\otimes\boldsymbol{Q}$ is an inner-product space by the bilinear form defined by $\langle x, y\rangle = x\cdot T_*y$ for $x, y\in K\otimes\boldsymbol{Q}$, where $\boldsymbol{Q}$ is the rational number field, and $K_1\otimes\boldsymbol{Q} + K_2\otimes\boldsymbol{Q}$ is a sub-inner-product space in $K\otimes\boldsymbol{Q}$. If $n_1n_2\neq 0$, then by the orthogonal decomposition theorem ([9]), $K\otimes\boldsymbol{Q}$ is isomorphic to an orthogonal sum $K_1\otimes\boldsymbol{Q} + K_2\otimes\boldsymbol{Q} + \mathrm{I}$, where I is a 1-dimensional inner-product space over $\boldsymbol{Q}$. This proves Assertion (1).

By Assertion (1) and Theorem 2, we obtain

$$\sigma(V, T)=\sigma(V_1, T_1)+\sigma(V_2, T_2)+\mathrm{c}(m_1, n_1)+\mathrm{c}(m_2, n_2)$$
$$-\mathrm{c}(m_1m_2, m_1n_2+m_2n_1)+\varepsilon,$$

where $\varepsilon=\pm 1$ if $n_1n_2\neq 0$, and $\varepsilon=0$ if $n_1n_2=0$. Now we assume that $n_1n_2\neq 0$. We must determine ε. Unfortunately we cannot determine ε geometrically. We use an algebraic condition as follows: By Theorem 2, $\sigma(V, T)\equiv\sigma(V_1, T_1)\equiv\sigma(V_2, T_2)\equiv 0 \pmod 8$. Hence $\mathrm{c}(m_1, n_1)+\mathrm{c}(m_2, n_2)-\mathrm{c}(m_1m_2, m_1n_2+m_2n_1)+\varepsilon\equiv 0 \pmod 8$. Now Hirzebruch ([6]) has shown the following formula

$$\mathrm{c}(p, q)\equiv 1-2\left(\frac{p}{q}\right)+q \pmod 8 \quad \text{for } q \text{ odd}<0,$$

where $\left(\frac{p}{q}\right)$ is the quadratic residue symbol. This formula implies the following

Assertion (2). *If* n *is odd,* $\mathrm{c}(m, n)\equiv n+\operatorname{sign} n \pmod 4$, *and if* n *is even,* $\mathrm{c}(m, n)\equiv m+n+(1+\operatorname{sign} n) \pmod 4$.

Proof. Since $-2\left(\frac{p}{q}\right)\equiv 2 \pmod 4$ and $\mathrm{c}(p, -q)=-\mathrm{c}(p, q)$, Hirzebruch's formula gives the first equation for n odd. For n even, use the relation $\mathrm{c}(m, n)=\mathrm{c}(m, m+n)-\operatorname{sign}(n(m+n))$ (note that m is odd).

By Assertion (2), a brief calculation shows that

$$-\mathrm{c}(m_1m_2, m_1n_2+m_2n_1)+\mathrm{c}(m_1, n_1)+\mathrm{c}(m_2, n_2)+\varepsilon$$
$$\equiv(\operatorname{sign} n_1+\operatorname{sign} n_2-\operatorname{sign}(m_1n_2+m_2n_1))+2+\varepsilon \pmod 4.$$

Since this must be $0 \pmod 4$ and both ε and $\operatorname{sign} n_1+\operatorname{sign} n_2-\operatorname{sign}(m_1n_2+m_2n_1)$ are ± 1 ($m_1m_2>0$), we obtain $\varepsilon=\operatorname{sign} n_1+\operatorname{sign} n_2-\operatorname{sign}(m_1n_2+m_2n_1)$. This completes the proof of Theorem 3 in the case with $k=2$. Now by the inductive calculation, we obtain the theorem.

REFERENCES

[1] W. BROWDER and G. R. LIVESAY: Fixed point free involutions on homotopy spheres, Bull. Amer. Math. Soc. **73** (1967), 242—245.

[2] J. EELLS and N. H. KUIPER: An invariant for certain smooth manifolds, Ann. Mat. Pura Appl. **60** (1963), 93—110.

[3] R. H. FOX: Free differential calculus III. Subgroups, Ann. Math. **64** (1956), 407—419.

[4] C. McA. GORDON: Knots, homology spheres, and contractible 4-manifolds, Topology **14** (1975), 151—172.

[5] F. HIRZEBRUCH: Involutionen auf Mannigfaltigkeiten, Proc. Conf. on Transformation Groups 148—166, New Orleans, 1967.

[6] F. Hirzebruch, W. D. Neumann and S. S. Koh: Differentiable Manifolds and Quadratic Forms, Lecture Notes in Pure and Appl. Math. **4**, Marcel Dekker, New York, 1971.

[7] J. Levine: Polynomial invariants of knots of codimension 2, Ann. Math. **84** (1966), 537—554.

[8] S. Lopez de Medrano: Involutions on Manifolds, Springer-Verlag, Berlin, 1971.

[9] J. Milnor and D. Husemoller: Symmetric Bilinear Forms, Springer-Verlag, Berlin, 1973.

[10] W. D. Neumann: S^1 actions and the α-invariant of their involutions, Bonner Math. Schriften **44** (1970).

[11] W. D. Neumann and F. Raymond: Seifert manifolds, Plumbing μ-invariant and orientation reversing maps, Lecture Notes in Math. **664**, 163—196, Springer-Verlag, Berlin, 1977.

[12] V. A. Rohlin: New results in the theory of four dimensional manifolds, Dokl. Akad. Nauk SSSR **84** (1952), 221—224 (in Russian).

Department of Mathematics
Okayama University

(*Received October 15, 1979*)